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On a lower and upper bound for the curvature of ellipses with more than two foci☆

A. Varga, Cs. Vincze*

University of Debrecen, Institute of Mathematics, P.O. Box 12, 4010 Debrecen, Hungary

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Abstract

Let a set of points in the Euclidean plane be given. We are going to investigate the levels of the function measuring the sum of distances from the elements of the pointset which are called foci. Levels with only one focus are circles. In case of two different points as foci they are ellipses in the usual sense. If the set of the foci consists of more than two points then we have the so-called polyellipses. In this paper we investigate them from the viewpoint of differential geometry. We give a lower and upper bound for the curvature involving explicit constants. They depend on the number of the foci, the rate of the level and the global minimum of the function measuring the sum of the distances. The minimizer will be characterized by a theorem due to E. Weiszfeld together with a new proof. Explicit examples will also be given. As an application we present a new proof for a theorem due to P. Erdős and I. Vincze. The result states that the approximation of a regular triangle by circumscribed polyellipses has an absolute error in the sense that there is no way to exceed it even if the number of the foci are arbitrary large.

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* Corresponding author. Tel.: +36 205907276.
E-mail address: csvincze@math.klte.hu (Cs. Vincze).

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1. Introduction

In this paper we are going to investigate the levels of a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ of the form

$$F(q) := \sum_{i=1}^{n} d(q, p_i).$$

They are called *polyellipses* with foci $p_1, \ldots, p_n$. The function is convex so generically has a unique minimum. When $n = 3$ this is the *Fermat-point* which can be given as the intersection of the circumcircles of the external equilateral triangles constructed on the sides of the triangle $p_1 p_2 p_3$. Define

$$c_\circ := \min\{F(q) | q \in \mathbb{R}^2\}, C := F^{-1}(c) \quad \text{and} \quad c_i := F(p_i)$$

for any indices $i = 1, \ldots, n$. Theorem 1 asserts

$$\frac{1}{2}\frac{c_1 + \cdots + c_n}{n-1} \leqslant c_\circ \leqslant \frac{c_1 + \cdots + c_n}{n}$$

and we also have

$$\left(\frac{c - c_\circ}{n}\right)^2 \pi \leqslant \mathrm{Area}(C) \leqslant \left(\frac{c + c_\circ}{n}\right)^2 \pi$$

in Corollary 1. According to Theorem 2 the curvature of the polyellipses satisfies

$$\kappa \leqslant \frac{c + c_\circ}{c - c_\circ} \sum_{i=1}^{n} \frac{1}{|c - c_i|}$$

and we also have

$$\frac{1}{\kappa} \leqslant \left(\frac{c}{2}\right)^6 \left(\frac{n}{2}\right)^3 \frac{n-1}{\mu^2[p_1, \ldots, p_n]} \sum_{i=1}^{n} \frac{1}{|c - c_i|}$$

for the reciprocal of the curvature in Theorem 3, where $\mu[p_1, \ldots, p_n]$ is the area of the convex hull of $p_1, \ldots, p_n$. One might hope that any convex closed curve Γ can be approximated by circumscribed polyellipses. Define

$$D(C, \Gamma) := \max_{p \in C} \min_{q \in \Gamma} d(p, q)$$

as the distance[1] of the curves. Theorem 4 says that for any regular triangle $\triangle$ there is a number $\varepsilon > 0$ such that for any circumscribed polyellipse C we have $D(C, \triangle) \geqslant \varepsilon$.

[1] The distance of two convex closed curves is usually defined as the maximum between $D(C, \Gamma)$ and $D(\Gamma, C)$. If the convex hull of C contains Γ than $D(C, \Gamma) \geqslant D(\Gamma, C)$. The proof is left as an exercise.

2. Preliminaries

In what follows the elements of the standard two-dimensional coordinate space $\mathbb{R}^2$ will be interpreted both as points and vectors as usual. According to the different interpretations we use the symbols $p, q, \ldots$ and $v, w, \ldots$ for the notation. Both the *length*

$$\|v\| := \sqrt{(v, v)}$$

or, in an equivalent terminology, the *norm* and the *distance*

$$d(p, q) := \|p - q\|$$

of the elements come from the canonical inner product

$$\langle v, w\rangle := v^1 w^1 + v^2 w^2$$

in the usual way.

Definition 1. Let $p_1, \ldots, p_n$ be not necessarily different points in the euclidean plane and consider the function $F: \mathbb{R}^2 \to \mathbb{R}$ defined by the formula

$$F(q) := \sum_{i=1}^{n} d(q, p_i). \tag{1}$$

The levels of the form $C := F^{-1}(c)$ are called *polyellipses* with the points $p_1, \ldots, p_n$ as foci. The *multiplicity* of the foci means that how many times they appear in the sum (1).

Using the triangle-inequality, it can be easily seen that the function F is *convex*. If the points $p_1, \ldots, p_n$ are not collinear then it is a *strictly convex* function. *Differentiability* is also clear everywhere except the points $p_1, \ldots, p_n$. It can be proved that any convex function has directional derivatives at *any* point in *any* direction; see e.g. [5, Theorem 30.5, p. 217]. If the function $d_1: \mathbb{R}^2 \to \mathbb{R}$ is defined by the formula

$$d_1(q) := d(q, p_1)$$

then the limit

$$d_1'(p_1, v) := \lim_{t \to 0^+} \frac{d_1(p_1 + tv) - d_1(p_1)}{t} = \|v\|$$

is just the directional derivative at the point p_1 in the direction v. Therefore,

$$F'(p_1, v) = \sum_{p_j = p_1} \|v\| + \sum_{p_j \neq p_1} \frac{1}{d(p_1, p_j)} \langle v, p_1 - p_j\rangle \tag{2}$$

and, of course, a similar formula holds for any indices $i = 1, \ldots, n$.

Definition 2. The element $w \in \mathbb{R}^2$ is called a *subgradient* of the function f at the point p if the inequality

$$\langle w, q - p\rangle \leqslant f(q) - f(p)$$

holds for any point $q \in \mathbb{R}^2$. The *subdifferential* of the function f is the set of its subgradients.

Remark 1. The subgradient involves a global property whereas the derivative has a local character. Nevertheless the convexity of the function allows us to describe the set of subgradients locally in terms of the directional derivative; for the following result we can refer to [1, Proposition 3.1.6, p. 35]. Since the proof is left as an exercise we present here a simple argumentation.

Lemma 1. *Let f be a convex function. The element w is a subgradient at the point p if and only if the inequality*

$$\langle w, v\rangle \leqslant f'(p, v)$$

holds for any vector $v \in \mathbb{R}^2$.

Proof. Suppose that w is a subgradient of the function f at the point p and let us choose the point q in the special form

$$q := p + tv,$$

where v is a nonzero vector and t is a positive real number. Then the relation

$$\langle w, v\rangle \leqslant \frac{f(p+tv) - f(p)}{t}$$

follows immediately from the definition of the subgradient. Taking the limit $t \to 0^+$ we have that

$$\langle w, v\rangle \leqslant f'(p, v)$$

as was to be stated. In order to see the converse statement let q be an arbitrary point and consider the line segment

$$c(t) := (1-t)p + tq$$

joining p and q. Since the function is convex, the formula

$$f \circ c(t) \leqslant (1-t) f(p) + t f(q)$$

holds for any parameter $0 \leqslant t \leqslant 1$. Therefore,

$$f'(p, v) = (f \circ c)'(0) = \lim_{t \to 0^+} \frac{f \circ c(t) - f \circ c(0)}{t} \leqslant f(q) - f(p),$$

where $v := q - p$. This means that

$$\langle w, q - p\rangle = \langle w, v\rangle \leqslant f'(p, v) \leqslant f(q) - f(p)$$

as was to be proved. $\square$

3. The problem of the minimizer

In what follows we give a new proof for the classical Weiszfeld's theorem characterizing the minimizer of the function F. Finding such a point where the global minimum is attained is crucial in the optimization problems formulated in terms of the polyellipses. They are often referred to as Fermat-problem, or Fermat–Steiner problem according to the original version: *given a triangle* $\triangle ABC$, *how can we find a point P for which* $PA + PB + PC$ *is minimal?* For the citation and further historical remarks see [4]. Here we propose another aspect of the motivations from the viewpoint of statistics: the minimizer can be interpreted as the *median* of the set of data $p_1, \ldots, p_n$.

Lemma 2. *Let* $K := [p_1, \ldots, p_n]$ *be the convex hull of the points* $p_1, \ldots, p_n$ *and consider the minimum*

$$m_\circ := \min\{F(q)|q \in K\}$$

attained at the point $q_\circ$ *of the convex compact set K. Then* $m_\circ$ *is a global minimum and, consequently, the point* $q_\circ$ *is a global minimizer.*

Proof. In order to see this global property we can use a standard nearest-point-type argumentation as follows. Let r be an arbitrary point in the plane which is not in K. Since K is a compact set, the function $d_r : K \to \mathbb{R}$ defined by the formula

$$d_r(q) := d(q, r)$$

attains its positive minimum at some point $q(r)$ contained in K. Let us consider the perpendicular bisector of the line segment $[q(r), r]$. The construction of the point $q(r)$ implies that the bisector separates the set K together with $p_1, \ldots, p_n$ from the point r. On the other hand, by the elements of the euclidean geometry, such a line divides the remaining points of the plane into two different open half-planes such that the points in the half-plane containing the set K together with $p_1, \ldots, p_n$ are closer to the endpoint $q(r)$ than to the other one. Therefore,

$$F(q(r)) < F(r)$$

which means that the function F takes its global minimum on the convex hull of the foci. □

Lemma 3. *If* $p_1, \ldots, p_n$ *are not collinear then the minimizer is uniquely determined.*

Proof. Since $p_1, \ldots, p_n$ are not collinear, the function F is strictly convex; it is easy to see that such a function can have at most one minimizer. □

Example 1. Let p_1, p_2, p_3 and p_4 be collinear points in the plane and suppose that p_1 and p_4 are the extremal points of their convex hull. For any point q in the line segment $[p_2, p_3]$ we have that

$$F(q) = d(p_1, p_4) + d(p_2, p_3)$$

and, as it can be easily seen, each of them is a minimizer.

Proposition 1 (*Weiszfeld [7]*). *Let $p_1, \dots, p_n$ be not necessarily different points in the euclidean plane. The point p_1 is a global minimizer of the function*

$$F(q) := d(q, p_1) + \cdots + d(q, p_n)$$

if and only if the length of the sum of the unit vectors going from p_1 to the p_j's whenever $p_1 \neq p_j$ is less or equal than the multiplicity.

Proof. A necessary and sufficient condition for a point to be the minimizer of a convex function is that the zero vector belongs to the set of the subgradients. According to Eq. (2) and Lemma 1 it is equivalent to the condition

$$0 \leqslant k + \frac{1}{\|v\|} \sum_{p_j \neq p_1} \frac{1}{d(p_1, p_j)} \langle v, p_1 - p_j \rangle,$$

where k is just the multiplicity of the point p_1. Since the right-hand side is constant along the rays emanating from the origin, it can be uniquely determined by the help of the evaluation along the unit circle. This means that there exists a global minimum of the expression. On the other hand, the Cauchy–Schwarz inequality shows that the minimum is attained if we substitute the vector

$$v := - \sum_{p_j \neq p_1} \frac{1}{d(p_1, p_j)} (p_1 - p_j),$$

which is just the sum of the unit vectors $v_1, \dots, v_{n-k}$ going from p_1 to the p_j's multiplied by -1. After substitution we have that the length of the sum is less or equal than k as was to be stated. □

Remark 2. It is clear that a similar statement can be formulated for any other point of the set $p_1, \dots, p_n$.

Definition 3. A minimizer is called *regular* if it does not belong to the set of the points $p_1, \dots, p_n$.

Proposition 2 (*Weiszfeld [7]*). *A necessary and sufficient condition for a point $p_\circ$ to be a regular minimizer is that the sum of the unit vectors going from $p_\circ$ to the p_j's is zero.*

Proof. As it is well-known from the standard calculus, if the function is convex and differentiable, then the vanishing of the partial derivatives at the point $p_\circ$ is a sufficient and necessary condition for $p_\circ$ to be a global minimizer. Since

$$D_1 F(p_\circ) = \sum_{j=1}^{n} \frac{1}{d(p_\circ, p_j)} (p_\circ^1 - p_j^1)$$

and

$$D_2 F(p_\circ) = \sum_{j=1}^{n} \frac{1}{d(p_\circ, p_j)} (p_\circ^2 - p_j^2),$$

the vanishing of the partial derivatives implies that the coordinates of the sum of the unit vectors going from $p_\circ$ to the p_j's vanish and vice versa. $\square$

Theorem 1. *For the minimum value $c_\circ$ of the function F we have*

$$\frac{1}{2}\frac{c_1+\cdots+c_n}{n-1}\leqslant c_\circ \leqslant \frac{c_1+\cdots+c_n}{n}, \tag{3}$$

where the constants $c_1, \ldots, c_n$ are defined by the formulas

$$c_1 := F(p_1), \ldots, c_n := F(p_n),$$

respectively.

Proof. The upper bound is trivial because

$$c_\circ \leqslant F\left(\frac{p_1+\cdots+p_n}{n}\right)\leqslant \frac{c_1+\cdots+c_n}{n}$$

according to the convexity. For the derivation of the lower bound we use the triangle inequality as follows:

$$c_1 = \sum_{j=2}^{n} d(p_1, p_j) \leqslant \sum_{j=2}^{n} d(p_1, p_\circ) + d(p_\circ, p_j)$$

$$= (n-1)d(p_1, p_\circ) + \sum_{j=2}^{n} d(p_\circ, p_j) = (n-2)d(p_1, p_\circ) + c_\circ$$

and a similar result holds for each of the further indices $i = 2, \ldots, n$. Taking the sum of these relations the lower bound

$$\frac{1}{2}\frac{c_1+\cdots+c_n}{n-1}\leqslant c_\circ$$

follows immediately. $\square$

Proposition 3. *The equality*

$$c_\circ = \frac{c_1+\cdots+c_n}{n}$$

holds if and only if the levels of the function F are ellipses in the usual sense or they are circles.

Proof. The relation

$$c_\circ \leqslant F\left(\frac{p_1+\cdots+p_n}{n}\right)\leqslant \frac{c_1+\cdots+c_n}{n}$$

implies that in case of the equality the foci must be collinear and each of them must be a minimizer. Suppose that $p_1, \ldots, p_m$ are different foci with the multiplicities $k_1, \ldots, k_m$.

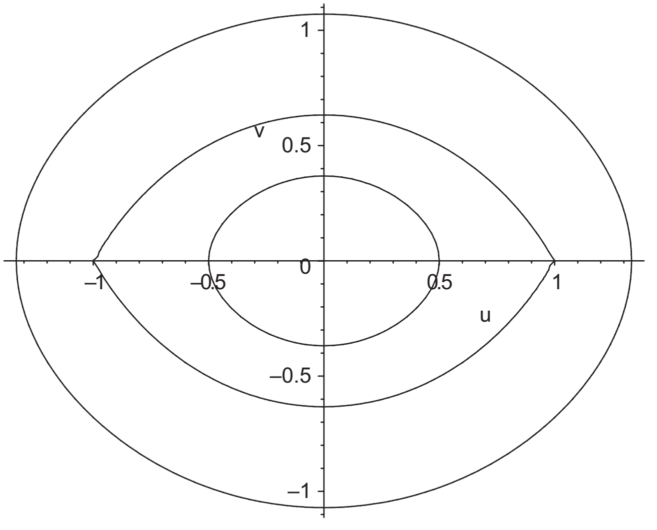

Fig. 1. Ellipses with three collinear foci in the plane.

Then $n = k_1 + \cdots + k_m$. Since the foci are collinear we can order them such that p_1 and p_m are the extremal points of their convex hull. Proposition 1 shows that

$$k_1 \geqslant k_2 + \cdots + k_m \quad \text{and} \quad k_m \geqslant k_1 + \cdots + k_{m-1}$$

and, consequently,

$$k_1 + k_m \geqslant k_1 + k_m + (k_2 + \cdots + k_{m-1}).$$

Since the inequalities have to reduce to the form

$$k_1 \geqslant k_2 \quad \text{and} \quad k_2 \geqslant k_1,$$

we have at most two different foci with the same multiplicity and the levels are ellipses in the usual sense or – as a special case – they are circles. $\square$

Remark 3. Fig. 1 shows how the lower bound can be attained in the non-trivial case of three different collinear foci.

4. Examples

The Fermat-problem. Given a triangle $\triangle ABC$, how can we find a point P for which the sum $PA + PB + PC$ is minimal? The answer is well-known with several types of solutions based on Viviani's theorem, Ptolemy's inequality or mechanical ideas, see e.g. [4]. The *Fermat-point* or, in an equivalent terminology, *the isogonic center* is the point from which the sides are seen at the angle 120°. It is obviously a necessary and sufficient condition for the triangle to have no angle which is greater or equal than 120°. If it has, then the Fermat-point is just the vertex where the critical angle occurs or, it is exceeded.

Ellipses with four foci. Suppose that the foci p_1, p_2, p_3 and p_4 are the vertex of a *convex quadrilateral* in the plane. Then the minimizer is just the intersection of the diagonals because the sum of the unit vectors going from this point to the p_i's is obviously zero; see Proposition 2. According to Proposition 1, the minimizer is just the vertex at the concave angle in case of a *concave deltoid*.

Exercise 1. Is it true for any concave quadrilateral in the plane?

Regular n-gons. Suppose that $n \geqslant 3$. If the foci form a regular n-gon then they are invariant under the rotations around the center with the magnitude $2k\pi/n$, where $k = 0, \dots, n-1$. The same is true for the minimizer. Since it is uniquely determined it must be the (common) center of the rotations.

Exercise 2. Explain how the symmetry about a line helps us locating the minimizer in case of a concave deltoid.

Parameterization. Let us consider the levels of the function F measuring the sum of the distances from the points

$$p_1 = (-1, 0), \quad p_2 = (0, 0) \quad \text{and} \quad p_3 = (1, 0).$$

It can be easily seen that $p_\circ = p_2$ is the minimizer and $c_\circ = 2$. Fig. 1 shows the levels in case of $c = \frac{5}{2}$, 3 and 4, respectively. Since the set of the foci are invariant under the reflections about their common line and the perpendicular bisector of the line segment $[p_1, p_3]$, it is enough to parameterize the part in the first quadrant of the coordinate plane.

Let us introduce the abbreviations

$$r_1 := d(p, p_1), \quad r := d(p, p_2), \quad r_3 := d(p, p_3),$$

where p is an arbitrary point except the origin. In terms of the polar angle α we have that $p = r(\cos\alpha, \sin\alpha)$ and the relations

$$\begin{aligned} r_1^2 &= r^2 + 1 + 2r\cos\alpha, \\ r_3^2 &= r^2 + 1 - 2r\cos\alpha \end{aligned} \tag{4}$$

can be immediately derived by the cosine-rule. If p is a point of the polyellipse defined by the formula $r_1 + r + r_3 = c$, then $r_1 + r_3 = c - r$. According to the equations in (4)

$$4r\cos\alpha = r_1^2 - r_3^2 = (r_1 - r_3)(r_1 + r_3) = (r_1 - r_3)(c - r)$$

and, consequently,

$$r_1 = r_3 + \frac{4r}{c - r}\cos\alpha.$$

Therefore,

$$c = r_1 + r + r_3 = 2r_3 + \frac{4r}{c-r}\cos\alpha + r,$$

which implies that

$$r_3 = \frac{1}{2}\left(c - r - \frac{4r}{c-r}\cos\alpha\right).$$

After substitution into the cosine-rule,

$$r\cos\alpha = \frac{c-r}{2}\sqrt{r^2 + 1 - \frac{(c-r)^2}{4}}.$$

Using the distance from the origin as the parameter, the function

$$x(r) := \frac{c-r}{2}\sqrt{r^2 + 1 - \frac{(c-r)^2}{4}} \tag{5}$$

gives the first coordinates of the points of the polyellipse. With Pythagoras' theorem

$$y(r) := \sqrt{\left(1 - \frac{(c-r)^2}{4}\right)\left(r^2 - \frac{(c-r)^2}{4}\right)}. \tag{6}$$

Since

$$2r \leqslant r_1 + r_3 = c - r \Rightarrow r \leqslant \frac{c}{3},$$

we have the interval

$$\frac{2}{3}\sqrt{c^2-3} - \frac{c}{3} \leqslant r \leqslant \min\left\{c-2, \frac{c}{3}\right\}$$

for the parameter r by requiring non-negative numbers under the square roots.

Remark 4. In case of $c = 3$ the curve contains the foci p_1 and p_3 as Fig. 1 shows.

Perimeter and area: With standard integral formulas such as

$$P = \int_a^b \sqrt{x'(r)^2 + y'(r)^2}\,\mathrm{d}r$$

and

$$A = -\int_a^b x(r)y'(r)\,\mathrm{d}r = \int_a^b x'(r)y(r)\,\mathrm{d}r$$

the perimeter and area of a domain bounded by a parameterized curve can be calculated. The date of the following table are computed by the computer-algebra system MAPLE:

Polyellipses	Perimeter	Area
$c = \frac{5}{2}$	2.7123	0.5645
$c = 3$	4.9603	1.7758
$c = 4$	7.5085	4.4032

5. On the curvature of polyellipses

In what follows we are going to investigate the polyellipses of the form $C := F^{-1}(c)$ from the viewpoint of differential geometry. According to the convexity of the function F these are convex curves in the plane. Recall that the plane curves are uniquely determined up to an isometry by the curvature function

$$\kappa := \frac{1}{\|\operatorname{grad} F\|}\left(\Delta F - \frac{1}{\|\operatorname{grad} F\|^2} F''(\operatorname{grad} F, \operatorname{grad} F)\right),$$

where ΔF and $\operatorname{grad} F$ denote the *Laplacian* and the *gradient* of the function F as usual. Since the foci are critical points, we shall suppose that the polyellipse under consideration doesn't contain any of them. Keeping the previous notations let $p_\circ$ be a minimizer with the minimum value $c_\circ := F(p_\circ)$ and let us define the constants $c_1, \dots, c_n$ as follows:

$$c_1 := F(p_1), \dots, c_n := F(p_n).$$

Remark 5. In [6] the authors proved that *if c is large enough then the polyellipse is contained between two concentric circles whose radii differ by an arbitrarily small amount*, Proposition 6, p. 247. In other words the curvature function goes to being identically zero under the limit $c \to \infty$. Here we are going to give not only a limit, but lower and upper bounds for the curvature involving explicit constants: the number of the foci, the rate of the level and the global minimum of the function F.

Lemma 4. *For any point $p \in C$ we have*

$$\frac{c - c_\circ}{n} \leqslant d(p, p_\circ) \leqslant \frac{c + c_\circ}{n}, \tag{7}$$

which means that the polyellipse is contained in the ring centered at the minimizer with the radii

$$r_1 := \frac{c - c_\circ}{n} \quad \textit{and} \quad r_2 := \frac{c + c_\circ}{n}.$$

Proof. Taking the sum with respect to the indices $i = 1, \dots, n$ both the upper and the lower bound can be derived by the help of the triangle inequalities

$$d(p, p_i) - d(p_i, p_\circ) \leqslant d(p, p_\circ) \leqslant d(p, p_i) + d(p_i, p_\circ),$$

where p is an arbitrary point of the polyellipse. $\square$

Remark 6. As a direct consequence of the previous result it follows that the convex hull of any polyellipse is a compact set; compactness and further convexity-topological properties in terms of the general notion of the norm are investigated in [3].

Corollary 1. *For the area of the domain bounded by a polyellipse we have the estimations*

$$\left(\frac{c-c_\circ}{n}\right)^2 \pi \leqslant A \leqslant \left(\frac{c+c_\circ}{n}\right)^2 \pi.$$

Lemma 5. *For any point $p \in C$ we have*

$$n\frac{c-c_\circ}{c+c_\circ} \leqslant \|\operatorname{grad} F_p\| \leqslant n. \tag{8}$$

Proof. From the definition of the subgradient it follows that if a convex function is differentiable at the point p, then

$$\langle \operatorname{grad} F_p, q-p\rangle \leqslant F(q)-F(p).$$

In case of $q = p_\circ$, the relation

$$c - c_\circ \leqslant \|\operatorname{grad} F_p\|\,\|p-p_\circ\|$$

is just the Cauchy–Schwarz inequality. According to Lemma 4, the polyellipse is contained in the circle centered at the minimizer with the radius

$$r = \frac{c+c_\circ}{n},$$

which gives the lower bound for the norm of the gradient at the point p. On the other hand, the gradient is just the sum of the unit vectors going from p to the p_i's. This means that the norm of this vector could not be greater than the number of the focuses as was to be stated. □

Remark 7. With a straightforward calculation we have that

$$\|\operatorname{grad} F_p\|^2 = n + 2\sum_{i<j} \cos\alpha_{ij},$$

where α_{ij} is the angle of the vectors $v_i := p_i - p$ and $v_j := p_j - p$.

Lemma 6. *For any point $p \in C$ we have*

$$n\sum_{i=1}^{n} \frac{1}{c+c_i} \leqslant \Delta F_p \leqslant n\sum_{i=1}^{n} \frac{1}{|c-c_i|}. \tag{9}$$

Proof. An easy straightforward calculation shows that

$$D_1 D_1 F_p = \sum_{i=1}^{n} \frac{1}{d^3(p,p_i)}(p^2 - p_i^2)^2$$

and a similar formula

$$D_2 D_2 F_p = \sum_{i=1}^{n} \frac{1}{d^3(p, p_i)} (p^1 - p_i^1)^2$$

holds in case of the second order derivatives. Therefore

$$\Delta F_p = \sum_{i=1}^{n} \frac{1}{d(p, p_i)} \tag{10}$$

for any regular point p in the plane. The proof follows a method of member by member. For any indices $i = 1, \ldots, n$:

$$\begin{aligned} |c - c_i| &\leqslant d(p, p_i) + \sum_{j \neq i} |d(p, p_j) - d(p_j, p_i)| \\ &\leqslant d(p, p_i) + (n-1) d(p, p_i) = n d(p, p_i). \end{aligned}$$

On the other hand,

$$\begin{aligned} c + c_i &= d(p, p_i) + \sum_{j \neq i} d(p, p_j) + d(p_j, p_i) \\ &\geqslant d(p, p_i) + (n-1) d(p, p_i) = n d(p, p_i). \end{aligned}$$

Therefore,

$$\frac{n}{c + c_i} \leqslant \frac{1}{d(p, p_i)} \leqslant \frac{n}{|c - c_i|}$$

which implies both the lower and upper bound for the Laplacian. □

Theorem 2. *The curvature function can be estimated by the formula*

$$\kappa_p \leqslant \frac{c + c_\circ}{c - c_\circ} \sum_{i=1}^{n} \frac{1}{|c - c_i|} \tag{11}$$

for any regular point $p \in C$.

Proof. Since the function F is convex, its second derivative function is positive semidefinite. Therefore,

$$\kappa_p \leqslant \frac{\Delta F_p}{\|\mathrm{grad} F_p\|},$$

which gives, by the help of the relations (8) and (9), the upper bound for the curvature function. □

Remark 8. Taking the limit $c \to \infty$ it can be easily seen that the curvature function goes to being identically zero.

In order to give a lower bound for the curvature we need the determinant of the matrix formed by the second-order derivatives. Since

$$D_1 D_2 F(p) = -\sum_{i=1}^{n} \frac{1}{d^3(p, p_i)}(p^1 - p_i^1)(p^2 - p_i^2)$$

we have that

$$\det D_i D_j F(p) = \sum_{i<j} \frac{1}{d^3(p, p_i) d^3(p, p_j)}((p^1 - p_i^1)(p^2 - p_j^2) - (p^1 - p_j^1)(p^2 - p_i^2))^2$$

which implies the formula

$$\det D_i D_j F(p) = 4 \sum_{i<j} \frac{1}{d^3(p, p_i) d^3(p, p_j)} \mu^2[p, p_i, p_j], \tag{12}$$

where μ means the area of the triangle spanned by the points p, p_i and p_j. Using the relation between the geometric and arithmetic means we have the estimation

$$\sqrt{d(p, p_i) d(p, p_j)} \leqslant \frac{d(p, p_i) + d(p, p_j)}{2} \leqslant \frac{c}{2}$$

and, consequently,

$$4 \left(\frac{2}{c}\right)^6 \sum_{i<j} \mu^2[p, p_i, p_j] \leqslant \det D_i D_j F(p).$$

Moreover, the square function is convex which implies that

$$\left(\sum_{i<j} \mu[p, p_i, p_j]\right)^2 \leqslant \binom{n}{2} \sum_{i<j} \mu^2[p, p_i, p_j].$$

Since for any point $p \in C$, the convex hull of the foci is obviously a subset of the union of the triangles $[p, p_i, p_j]$,

$$\mu[p_1, \ldots, p_n] \leqslant \sum_{i<j} \mu[p, p_i, p_j]$$

and we have just proved the following result.

Lemma 7. *For any regular point $p \in C$*

$$8 \left(\frac{2}{c}\right)^6 \frac{1}{n(n-1)} \mu^2[p_1, \ldots, p_n] \leqslant \det D_i D_j F(p). \tag{13}$$

Remark 9. As the previous result shows if the foci are not collinear then the second order partial derivatives form the coefficients of a positive definite bilinear form.

Theorem 3. *Suppose that the foci are not collinear; the reciprocal of the curvature function can be estimated by the formula*

$$\frac{1}{\kappa_p} \leqslant \left(\frac{c}{2}\right)^6 \left(\frac{n}{2}\right)^3 \frac{n-1}{\mu^2[p_1, \ldots, p_n]} \sum_{i=1}^{n} \frac{1}{|c-c_i|} \tag{14}$$

for any regular point $p \in C$.

Proof. Let λ_1 and λ_2 be the eigenvalues of the matrix consisting of the second order partial derivatives at the point p and suppose that they are written in nonincreasing order $\lambda_1 \geqslant \lambda_2$. Since they are just the solutions of the characteristic equation

$$\lambda^2 - \lambda \Delta F_p + \det D_i D_j F(p) = 0,$$

we have that

$$\det D_i D_j F(p) \leqslant \lambda_2^2 + \det D_i D_j F(p) = \lambda_2 \Delta F_p. \tag{15}$$

On the other hand, the eigenvalue λ_1 is the maximum of the second derivative

$$F''(p)\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

on the unit circle, which means that

$$0 \leqslant \lambda_1 - \frac{1}{\|\operatorname{grad} F_p\|^2} F''(\operatorname{grad} F_p, \operatorname{grad} F_p).$$

Therefore,

$$\lambda_2 \leqslant \lambda_2 + \lambda_1 - \frac{1}{\|\operatorname{grad} F_p\|^2} F''(\operatorname{grad} F_p, \operatorname{grad} F_p) = \kappa_p \|\operatorname{grad} F_p\| \tag{16}$$

because the Laplacian is just the sum of the eigenvalues. Formulas (15) and (16) shows that

$$\frac{1}{\kappa_p} \leqslant \|\operatorname{grad} F_p\| \frac{\Delta F_p}{\det D_i D_j F(p)},$$

where all of the terms can be estimated by (8), (9) and (13). □

Corollary 2. *For the perimeter of the domain bounded by a polyellipse we have the estimation*

$$P \leqslant 2\pi \left(\frac{c}{2}\right)^6 \left(\frac{n}{2}\right)^3 \frac{n-1}{\mu^2[p_1, \ldots, p_n]} \sum_{i=1}^{n} \frac{1}{|c-c_i|}.$$

Proof. Using the Hopf Umlaufsatz we have that

$$\int_0^P \kappa(s)\, \mathrm{d}s = 2\pi,$$

where $\kappa(s)$ is the curvature function of the polyellipse parameterized by arclength. The upper bound (14) for the reciprocal of the curvature function gives a lower bound for the curvature. With such an estimation one can easily derive the upper bound for the perimeter. □

Exercise 3. Using (11) find a lower bound for the perimeter of the domain bounded by a polyellipse.

6. On the approximation of a regular triangle by circumscribed polyellipses

The problem whether all of convex plane curves can be arbitrarily approximated by polyellipses under a sufficiently large number of the foci was posed by Endre Vázsonyi. In case of a circle we have an approximating process by polyellipses such that each curve contains all of its foci forming a regular circumscribed n-gon under the limit $n \to \infty$; see Section 3.

Let r be a positive real parameter and consider the family of polyellipses determined by the equation

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2} + \sqrt{(x-r)^2 + y^2} = 2 + \sqrt{r^2+1}$$

as in [2]. It can be easily seen that each polyellipse C_r has the foci

$$p_1 := (0, 1), \quad p_2 := (0, -1), \quad p_3 := (r, 0)$$

and each of them contains the points p_1 and p_2. Taking the limit $r \to \infty$, the formula

$$\sqrt{x^2 + (y+1)^2} + \sqrt{x^2 + (y-1)^2} = 2 + x$$

determines a curve C_∞ in the plane such that it contains the line segment $[p_1, p_2]$; see Fig. 2.

The "limit curve" can be arbitrarily approximated by circumscribed polyellipses with three foci. More precisely, the distance

$$D(C_r, C_\infty) := \max_{p \in C_r} \min_{q \in C_\infty} d(p, q)$$

of the curves goes to being zero under the limit $r \to \infty$; see Fig. 3.

In [2] the authors proved that there is no way to reach a regular triangle by the help of a similar process even if the increase of the number of foci is allowed. In what follows we present a new proof for this theorem using the tools of the differential geometry of plane curves.

Theorem 4. *The approximation of a regular triangle by circumscribed polyellipses always has an absolute error which cannot be exceeded even if the number of foci are arbitrary large.*

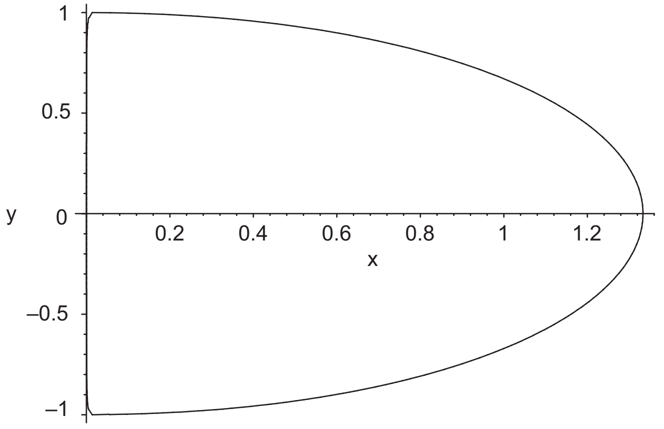

Fig. 2. The limit curve under $r \to \infty$.

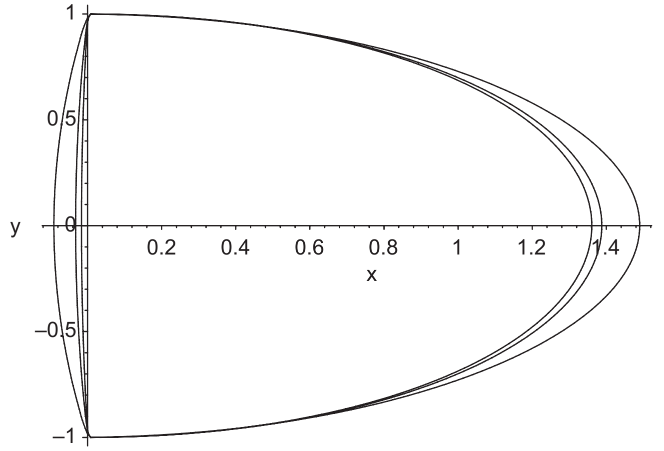

Fig. 3. Polyellipses with the parameters $r = 5$, 15 and 30.

Proof. Let $\triangle$ be a regular triangle in the plane and suppose, in contrary, that there exists a sequence $\mathscr{E}_1, \ldots, \mathscr{E}_n, \ldots$ of circumscribed polyellipses such that

$$\lim_{n \to \infty} D(\mathscr{E}_n, \triangle) = 0.$$

We use the symbol H for notating the subgroup of isometries which leave the triangle $\triangle$ invariant, i.e. H consists of the identity, the reflections about the heights and the rotations around the center with angles $\pm 120°$, respectively. $\square$

Lemma 8. *Let $\mathscr{E} := F^{-1}(c)$ be a circumscribed polyellipse with the foci $p_1, \ldots, p_n$. Then*

$$D(\hat{\mathscr{E}}, \triangle) \leqslant D(\mathscr{E}, \triangle),$$

where $\hat{\mathscr{E}}$ is the polyellipse passing through all of the vertices A, B and C of the triangle with the elements of the set

$$G := \{f(p_i)|i = 1, \dots n \text{ and } f \in H\}$$

as foci.

Proof. First of all note that the role of the vertices A, B and C are entirely symmetric as the formulas

$$\sum_{f\in H} d(A, f(p_i)) = \sum_{f\in H} d(B, f(p_i)) = \sum_{f\in H} d(C, f(p_i)), \quad i = 1, \dots, n$$

shows: if $f_*(A) = B$ for some $f_* \in H$ then

$$\begin{aligned}\sum_{f\in H} d(A, f(p_i)) &= \sum_{f\in H} d(f_*(A), f_* \circ f(p_i)) = \sum_{f\in H} d(B, f_* \circ f(p_i)) \\ &= \sum_{f\in H} d(B, f(p_i))\end{aligned}$$

because $f_* \circ f$ runs through the elements of H as f does. Therefore,

$$\hat{\mathscr{E}} := \hat{F}^{-1}(\hat{c}), \quad \text{where } \hat{c} := \sum_{f\in H} d(A, f(p_1)) + \cdots + d(A, f(p_n)).$$

Let now $f \in H$ be fixed for a moment and consider the polyellipse $f(\mathscr{E})$ with the foci $f(p_1), \dots, f(p_n)$ such that the sum of distances from the foci is just c. $f(\mathscr{E})$ is the image of $\mathscr{E}$ under f. The symmetry implies that $\triangle \subset f(\mathscr{E})$ and, consequently,

$$d(A, f(p_1)) + \cdots + d(A, f(p_n)) \leqslant c.$$

Since such a formula holds for both B and C, we have that $\hat{c} \leqslant 6c$. Consider the union of the convex hull of the sets $f(\mathscr{E})$, where f runs through the elements of H. The points of the polyellipse $\hat{\mathscr{E}}$ together with the points of the triangle are contained in the union because for any outer point q in the plane

$$d(q, f(p_1)) + \cdots + d(q, f(p_n)) > c$$

for all $f \in H$. Therefore,

$$\hat{F}(q) = \sum_{f\in H} d(q, f(p_1)) + \cdots + d(q, f(p_n)) > 6c \geqslant \hat{c}.$$

On the other hand, the boundary Γ of the union is not farther from the triangle than the polyellipse $\hat{\mathscr{E}}$. Indeed, the distance of the curves are determined by the distances of the points running through them and, according to the constructing process, there is no new

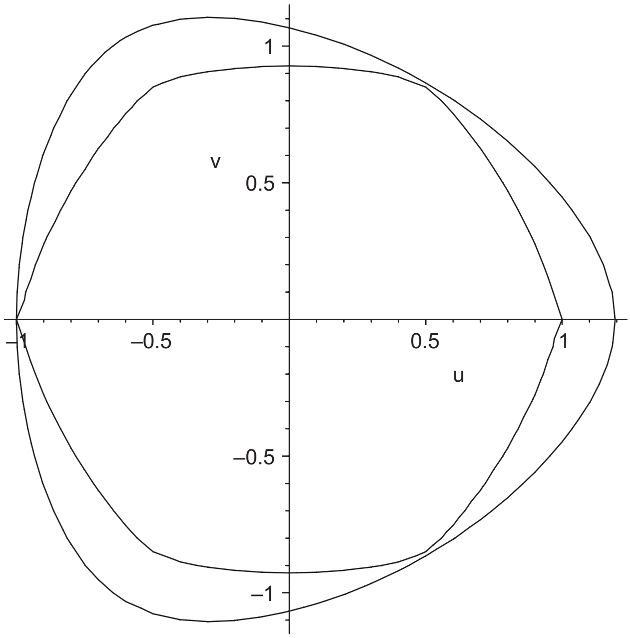

Fig. 4. The pair of curves $\mathscr{E}_i$ and $\hat{\mathscr{E}}_i$.

one for the beginning. Therefore,

$$D(\hat{\mathscr{E}}, \triangle) \leqslant D(\Gamma, \triangle) \leqslant D(\mathscr{E}, \triangle)$$

as was to be stated. $\square$

In view of Lemma 8 we can suppose that the sequence of polyellipses $\mathscr{E}_1, \ldots, \mathscr{E}_n, \ldots$ consists of curves all of whose foci form an invariant set under the elements of the group H. Let $q_n \neq C$ be the common point of the polyellipse $\mathscr{E}_n$ and the perpendicular bisector of the side AB. The limit of the q_i's is just the mid-point of AB because the curves are convex and they go to the triangle in the sense of the Blaschke metric. Note that the euclidean distance δ_i between the point q_i and the mid-point of AB is just the Blaschke-distance. From the viewpoint of differential geometry we have to consider two different cases: the point q_i belongs to the set of the foci or not. If it does then we ignore this point together with the foci of the form $f(q_i)$, where $f \in H$. In this case we substitute $\mathscr{E}_i$ with a new polyellipse $\hat{\mathscr{E}}_i$ as follows:

(a) the foci are the rest of those of $\mathscr{E}_i$,
(b) the curve contains all of ex-foci $f(q_i)$ as Fig. 4 shows.

Let k_i be the multiplicity of the point q_i as the focus of the polyellipse $\mathscr{E}_i$. Then the sum of distances from the rest of the focuses must be

$$\hat{c}_i = c_i - k_i \sum_{f \in H} d(q_i, f(q_i)) = c_i - 4k_i \left(\frac{1}{2} d(A, B) + \sqrt{3}\delta_i \right).$$

The last equation can be easily derived with using similar triangles:

$$d(q_i, f(q_i)) = \frac{\delta_i + \frac{1}{3}m}{\frac{2}{3}m} d(A, B),$$

where m is the height and $f(q_i) \neq q_i$ (it happens four times). On the other hand,

$$\begin{aligned}
&k_i \sum_{f \in H} d(A, f(q_i)) \\
&\quad \geqslant k_i \left(\frac{1}{2} d(A, B) + \frac{1}{2} d(A, B) + \frac{1}{2} d(A, B) + \frac{1}{2} d(A, B) + m + m \right) \\
&\quad \geqslant 4k_i \left(\frac{1}{2} d(A, B) + \frac{\sqrt{3}}{4} d(A, B) \right) \geqslant 4k_i \left(\frac{1}{2} d(A, B) + \sqrt{3}\delta_i \right)
\end{aligned}$$

provided that δ_i is small enough. This means that all of the vertices is in the interior of the convex hull of $\hat{\mathscr{E}}_i$ as Fig. 4 shows: the foci are the vertices of a regular hexagon inscribed in the unit circle centered at the origin.

The role of the polyellipses $\hat{\mathscr{E}}_1, \ldots, \hat{\mathscr{E}}_n, \ldots$ is to present a sequence consisting of curves all of whose foci form an invariant set under the elements of the group H and the curvature goes to being zero at the q_i's; see Exercise 4. It will be proved that the curvature function is uniformly bounded from below which obviously gives a contradiction.

Exercise 4. Prove that the curvature radius at the point q_i is greater than the radius r_i of the circle passing through the points A, B and q_i. Use the relation

$$(r_i - \delta_i)^2 = r_i^2 - \tfrac{1}{4} d^2(A, B)$$

to conclude that $\lim_{i \to \infty} r_i = \infty$.

Lemma 9. *Let R be the radius of the circumscribed circle of the triangle and suppose that the point $q := q_n$ lies in the interior of the circle for a sufficiently large integer n. Then*

$$\|\operatorname{grad} F_q\| \leqslant \sum_{i=1}^{n} \frac{24R}{R + d(o, p_i)},$$

where o is the center of the triangle, and the set G of the foci are generated by the points $p_1, \ldots, p_n$.

Proof. As we have seen above, the gradient at the point q is just the sum of the unit vectors going from q to the foci:

$$\|\operatorname{grad} F_q\| = \left\| \sum_{i=1}^{n} \sum_{f \in H} \frac{1}{d(q, f(p_i))} (q - f(p_i)) \right\|.$$

Let k_o be the multiplicity of the center if it is one of the foci; otherwise $k_o := 0$. Then

$$\|\operatorname{grad} F_q\| \leqslant 6k_o + \left\| \sum_{p_i \neq o} \sum_{f \in H} \frac{1}{d(q, f(p_i))}(q - f(p_i)) \right\|,$$

because $f(o) = o$ for any element of H. Since o is the minimizer of F

$$\sum_{p_i \neq o} \sum_{f \in H} \frac{1}{d(o, f(p_i))}(o - f(p_i)) = 0$$

and we have that the norm of the gradient at the point q is less or equal than

$$6k_o + \sum_{p_i \neq o} \sum_{f \in H} \left\| \frac{1}{d(q, f(p_i))}(q - f(p_i)) - \frac{1}{d(o, f(p_i))}(o - f(p_i)) \right\|.$$

The estimation

$$6k_o \leqslant \sum_{p_i = o} \frac{24R}{R + d(p_i, o)} = 24k_o$$

is trivial. From now on we suppose that $p_i \neq o$. In order to estimate the norm of the difference of the unit vectors

$$v_i := \frac{1}{d(q, f(p_i))}(q - f(p_i))$$

and

$$w_i := \frac{1}{d(o, f(p_i))}(o - f(p_i)) = \frac{1}{d(o, p_i)}(o - f(p_i)),$$

consider first of all the case when the focus p_i together with all of the elements of the set

$$G_i := \{f(p_i) | f \in H\}$$

is in the interior of the circumscribed circle of the triangle. Since the norm of the difference of unit vectors is less or equal than 2 it follows that

$$\frac{4R}{R + d(o, p_i)} \geqslant 2 \geqslant \|v_i - w_i\|. \tag{17}$$

The only task is to prove (17) for the foci having distances from the origin more or equal than the radius of the circle. From the triangle spanned by the vectors v_i and w_i with the same (unit) length we have that

$$\|v_i - w_i\|^2 = 2(1 - \cos \alpha_i) = 4 \sin^2 \frac{\alpha_i}{2} \leqslant 4 \sin^2 \alpha_i \Rightarrow \|v_i - w_i\| \leqslant 2 \sin \alpha_i$$

because the angle α_i of the unit vectors v_i and w_i is less than 90°. As the sine-rule shows

$$d(o, p_i) \sin \alpha_i = d(o, f(p_i)) \sin \alpha_i = d(o, q) \sin(\text{some angle}) \leqslant d(o, q) \leqslant R.$$

Therefore,

$$2R\sin\alpha_i + 2d(o, p_i)\sin\alpha_i \leqslant 2R\sin\alpha_i + 2R \leqslant 4R$$

and, consequently,

$$2\sin\alpha_i \leqslant \frac{4R}{R + d(o, p_i)} \Rightarrow \|v_i - w_i\| \leqslant \frac{4R}{R + d(o, p_i)}$$

as was to be proved. Taking the sum with respect to $f \in H$ we have the upper bound for the gradient immediately. □

Lemma 10. *If $q := q_n$ lies in the interior of the circumscribed circle of the triangle, then*

$$F''(q)(v^*, v^*) \geqslant \frac{1}{4}\sum_{i=1}^{n} \frac{1}{R + d(o, p_i)},$$

where v^* is a parallel unit vector to the side AB of the triangle.

Proof. Suppose that the center of the triangle coincides with the origin and, after rotating around the center if necessary, let the vector v^* be written in the form

$$v^* := (0, 1).$$

The formula for the second order partial derivative D_2D_2F shows that we have to estimate the sum of type

$$h(q, p_i) := \sum_{f \in H} \frac{1}{d^3(q, f(p_i))}(q^1 - f^1(p_i))^2$$

n times. By the triangle inequality,

$$d(q, f(p_i)) \leqslant d(q, o) + d(o, f(p_i)) \leqslant R + d(o, p_i)$$

and, consequently,

$$\frac{1}{R + d(o, p_i)} \sum_{f \in H} \frac{1}{d^2(q, f(p_i))}(q^1 - f^1(p_i))^2 \leqslant h(q, p_i).$$

The case of $p_i = o$ is trivial because

$$\frac{1}{d^2(q, f(o))}(q^1 - f^1(o))^2 = 1 \geqslant \frac{1}{4}$$

for any $f \in H$. If $p_i \neq o$ then for some isometry $f \in H$ the polar angle of the point $f(p_i)$ must be between 120° and 240°. Therefore

$$\frac{1}{4} = \cos^2 60° \leqslant \frac{1}{d^2(q, f(p_i))}(q^1 - f^1(p_i))^2;$$

note that it happens at least two times. Taking the sum with respect to the indices $i = 1, \ldots n$, the statement follows immediately. □

Now we are in the position to finish the proof of the theorem. According to the symmetry of the polyellipses, v^* can be considered as the tangent unit vector at the point $q = q_n$. On the other hand, (1, 0) is the parallel unit vector to $\operatorname{grad} F_q$. Therefore

$$\kappa(q) = \frac{1}{\|\operatorname{grad} F_q\|}(\Delta F_q - D_1 D_1 F_q) = \frac{1}{\|\operatorname{grad} F_q\|} D_2 D_2 F_q$$
$$= \frac{1}{\|\operatorname{grad} F_q\|} F''(q)(v^*, v^*) \geqslant \frac{1}{96R} \tag{18}$$

which is a contradiction. □

Remark 10. As Fig. 4 shows, Eq. (18) involves a global minimum for the curvature along the whole arc of the polyellipses because of the symmetry. The method presented in the proof can be used for the estimation of the curvature in all of the cases when the set of the foci shows invariance under some isometries.

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